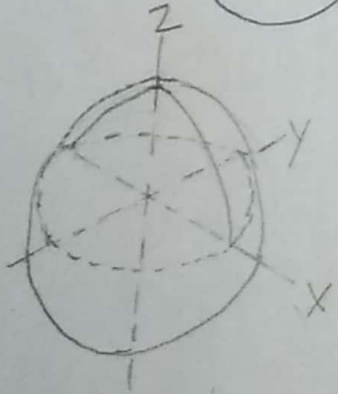
Unit-4 Mass M.I and Virtual Work

* Determine the mass M.I of a sphere of radius (r) and mass (m) about an axis passing through its origin. Assume density of sphere is constant (or)
Determine mass M.I of sphere about its diametral axis

A). Consider a sphere of radius (R) and mass (m). Suppose we cut a thin circular disc of radius (r) and infinitesimal thickness dz at a distance (z) from X-Y plane (Cross section of the sphere) then its mass is given as $dm = \rho \pi r^2 dz$.

Therefore, mass M.I of disc about vertical Z-axis is given as

$$(dI_{zz})_{mass} = \frac{dm\, r^2}{2} = \rho\left(\frac{\pi r^4}{2}\right)dz$$

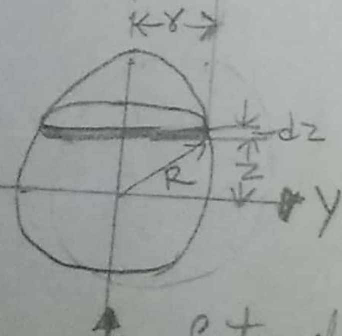


Integrating the above expression b/w limits we get mass M.I of sphere about Z-axis as

$$I_{zz} = \int_{-R}^{R} \rho\left(\frac{\pi r^4}{2}\right)dz$$

Since R is radius of sphere

$$z^2 + r^2 = R^2 \quad | \quad r^2 = R^2 - z^2$$



$$I_{zz} = \int_{-R}^{R} \rho\pi \frac{(R^2-z^2)^2}{2} dz$$

Section of plane $= \frac{\rho\pi}{2}\int_{-R}^{R}(R^4+z^4-2R^2z^2)dz$

$$I_{zz} = \frac{\rho\pi}{2}\left[R^4 z + \frac{z^5}{5} - 2R^2\frac{z^3}{3}\right]_{-R}^{R} = \frac{8}{15}\rho\pi R^5$$

Volume of sphere $= \frac{4}{3}\pi R^3$

$$M = \rho V = \frac{4}{3}\rho\pi R^3$$

$$I_{zz} = \frac{8}{15}\rho\pi R^5 = \frac{8}{15}\rho\pi R^3 R^2 = \frac{2}{5}\left(\frac{4}{3}\rho\pi R^3\right)R^2$$

$$= \frac{2}{5}MR^2$$

Due to symmetry, mass M.I remains same for any axis passing through origin $I = \frac{2}{5}MR^2$

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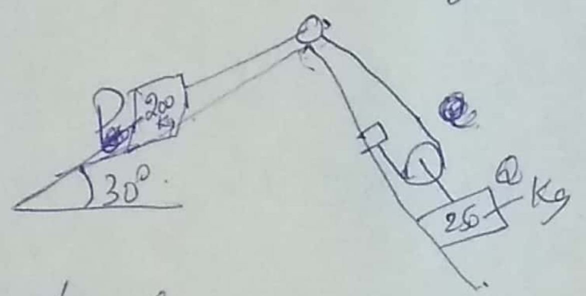
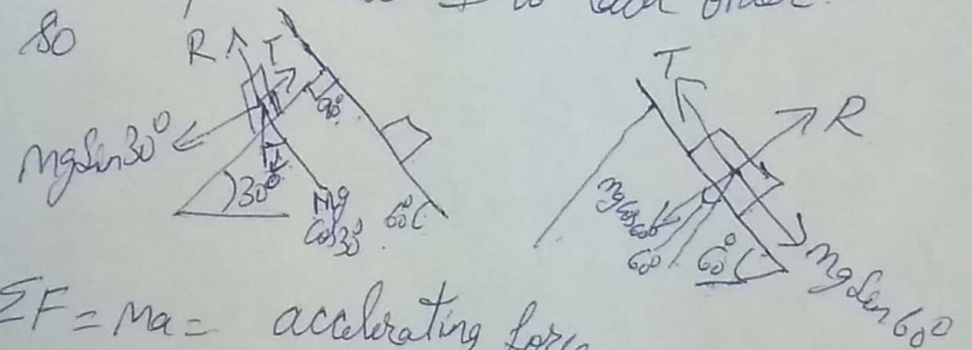
* Find tension in string as shown in fig



A) Here two planes are $\perp$ to each other.

So



$\Sigma F = Ma =$ accelerating force

T is taken as +ve in 200 kg body (P), $mg\sin 30$ is taken as −ve

$T - mg\sin 30 = M_1 a$ —(1)

as 200 kg body moving upwards

body of 250 kg (Q)

T is taken as −ve, $M_2 g\sin 60$ is taken as +ve

as Q is moving downwards

$M_2 g\sin 60 - T = M_2 a$ —(2)

Solving ① & ② eq.

$T - M_1 g \sin 30 = M_1 a$

$-T + M_2 g \sin 60 = M_2 a$

$g(M_2 \sin 60 + M_1 \sin 30^\circ) = a(M_2 + M_1)$

$9.81\left(250 \times \frac{\sqrt{3}}{2} + 200 \times \frac{1}{2}\right) = a(200 + 250)$

$a = 6.89\, m/s^2$

$T = 150 \times 6.89 + 150 \times 9.81 \sin 30^\circ$ (Using ① eq)

$= 1769.25 N$

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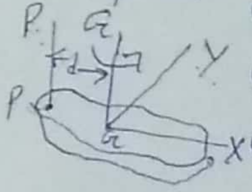
* Define mass moment of Inertia and explain the transfer formula for mass M.I ?

A) Mass moment of Inertia of a solid measures the solid ability to resist (oppose) changes in rotational speed about specific axis



<u>Transfer Formula for mass moment of Inertia:</u>

<u>Theorem:</u> The mass moment of inertia of a body about an axis at a distance (d) and parallel to the centroidal (G) axis is equal to the sum of the moment of inertia about the centroidal axis and product of mass and square of ⊥ distance b/w the parallel axes.

~~I_G = MoI about cen~~

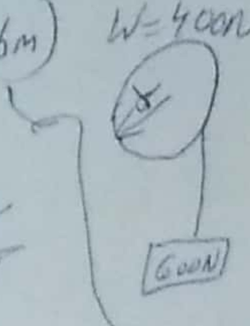
I_G = mass M.I about Center of mass, G

I_P = mass M.I about an axis passing through P (parallel to G axes)

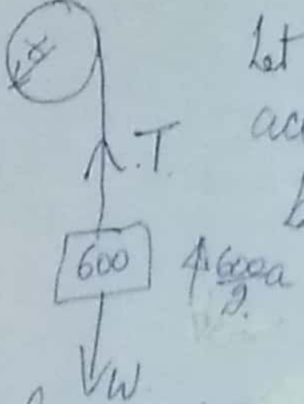
M, d^2 = mass of body, Square of ⊥ distance b/w axes.

$$\boxed{I_P = I_G + Md^2}$$

* A pulley of 400 N weight has a radius of 0.6m. A block of 600N is suspended by a tight rope wound round the pulley, the other end being attached to the pulley. Determine the resulting acceleration of the weight and tension in the rope ($r = 0.6m$)



A)



Let a be resulting acceleration and T be Tension in rope. Hence angular acceleration of pulley ($a = \alpha r$)

$$\alpha = \frac{a}{r} = \frac{a}{0.6}$$

$$\alpha = 1.667a \ rad/s^2$$

An inertia force of $\left(\frac{600a}{g}\right)$ may be considered and dynamic (when body is in motion) condition can be

$$T + \frac{600a}{9.81} = 600$$

$$T = 600 - \frac{600a}{9.81}$$

For Kinematic equation of pulley

The rotational moment is equal to product of mass MI and angular acceleration. $M = \cancel{F} I\alpha$

$$T \times r = I\alpha$$

$$T \times 0.6 = I\,(1.667)a$$

we know that MI $I = \frac{mr^2}{2}$ for pulley

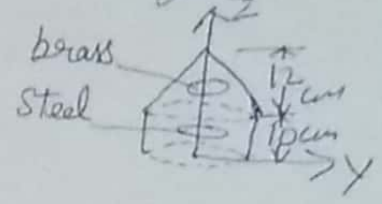
$$m = \frac{400}{9.81}$$

$$T = \frac{\frac{400}{9.81} \times \frac{0.6^2}{2}\,(1.667)\,a}{0.6}$$

$$600 - \frac{600a}{9.81} = \frac{200}{9.81}a$$

$a = 7.358 \ m/s^2$ | $T = \frac{200}{9.81} \times 7.35 = 150N$

* Determine Centre of mass of a Composite body formed by placing a brass Cone with a base diameter of 8cm and 12cm height over a steel cylinder of same diameter and height of 10 cm Density of steel is 7850 Kg/m³ and that of brass is 8650 Kg/m³.



Ans. A) Brass

$r_1 = 4\,cm$, $h_1 = 12\,cm$
$= 0.04\,m$, $= 0.12\,m$.

$\rho_1 = 8650\ kg/m^3$

For steel

$r_2 = 0.04\,m$, $h_2 = 10\,cm = 0.1\,m$, $\rho_2 = 7850\ kg/m^3$

Volume of Cone (brass) $= \frac{1}{3}\pi R_1^2 H_1 = \frac{1}{3}\pi \times 0.04^2 \times 0.12$

$V_1 = 2.01 \times 10^{-4}\ m^3$

Mass $(M_1) = \rho_1 V_1 = 8650 \times 2.01 \times 10^{-4} = 1.738\ Kg$

Volume of Cylinder (steel) $= \pi r_2^2 H_2 = \pi \times 0.04^2 \times 0.1$
$= 5.026 \times 10^{-4}\ m^3$

Mass $(M_2) = \rho_2 V_2 = 7850 \times 5.026 \times 10^{-4} = 3.94\ Kg$

Mass M.I of Cone (brass)

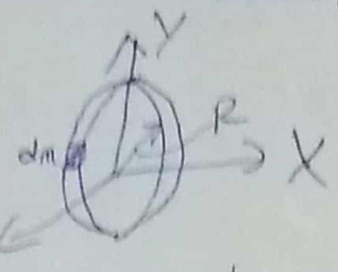
$I_{zz_1} = \frac{3}{10} M_1 R_1^2 = \frac{3}{10} \times 1.738 \times 0.04^2$
$= 8.34 \times 10^{-4}\ Kg\,m^2$

Mass M.I of Steel (Cylinder)

$I_{zz_2} = \frac{M_2 R_2^2}{2} = \frac{3.94 \times 0.04^2}{2} = 3.152 \times 10^{-3}\ Kg\,m^2$

$I_{zz} = I_{zz_1} + I_{zz_2} = 8.34 \times 10^{-4} + 3.152 \times 10^{-3}$
$= 3.986 \times 10^{-3}\ Kg\,m^2$

* Determine the mass M.I of Circular (or) hoop ring of mass (M) and radius (R) about Centroidal axes



A). Let the ring be in X-Y plane. If we take an infinitesimally small element of mass (dm) then its mass M.I about Z axis is

$$dI_{zz} = dm\,R^2$$

mass M.I of entire ring about Z axis is

$$I_{zz} = \int R^2\,dM$$

Since every such infinitesimal element lies at same radial distance R,

$$I_{zz} = R^2 \int dm = MR^2$$

Hence, mass M.I about X and Y axes are

$$I_{xx} = I_{yy} = MR^2/2.$$

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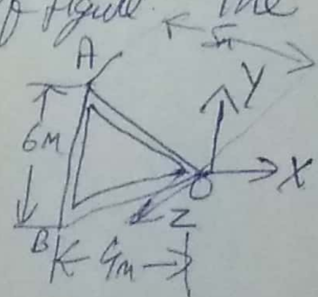
* A uniform steel rod is bent into shape of an isosceles triangle. Compute the radius of gyration about axis through O ⊥ to plane of figure. The total mass of steel rod is 10kg.



A) From figure

$$\overline{OB} = \sqrt{4^2 + \left(\frac{6}{2}\right)^2} = 5m$$

Total length of rod $= \overline{OA} + \overline{OB} + \overline{BA} = 5+5+6 = 16m$.

Total mass of steel rod is 10kg

mass of each side is $\boxed{\dfrac{M_{side}}{M_{total}} = \dfrac{L_{side}}{L_{total}}}$

$$M_{OA} = 10 \times \frac{5}{16}$$

$$= 3.125 Kg$$

$M_{AB} = 10 \times \dfrac{6}{16} = 3.75 Kg$ / $M_{OB} = 10\left(\dfrac{5}{16}\right) = 3.125 Kg$

Calculate mass M.I for portion OA, AB, OB

portion OA Mass M.I about Centroidal axes. Z-axis (I_{zzG})

$$(I_{zzG})_{OA} = \frac{M_{OA} L_{OA}^2}{12} = \frac{3.125 \times 5^2}{12} = 6.51 Kgm^2$$

mass M.I of OA about Z-axis of Composite rod is

$$(I_{zz})_{OA} = (I_{zzG})_{OA} + (M_{OA})(d_{OA})^2 \quad d_{OA} = \frac{5}{2} = 2.5$$

$$(I_{zz})_{OA} = 6.51 + 3.125(2.5)^2 = 26.04 Kgm^2$$

For portion AB

$$(I_{zzG})_{AB} = \frac{M_{AB} L_{AB}^2}{12} = \frac{3.75 \times 6^2}{12}$$

d_{AB} = distance from AB rod side to origin $= 4m$.

$$= 11.25 Kgm^2$$

$$(I_{zz})_{AB} = (I_{zzG}) + M_{AB}(d_{AB})^2$$

$$= 11.25 + (3.75 \times 4^2) = 71.25 Kgm^2$$

For portion BO

$$(I_{zzG})_{OB} = \frac{M_{OB} L_{OB}^2}{12} = \frac{3.125 \times (5)^2}{12} = 6.51 Kgm^2$$

$(I_{zz})_{OB} = (I_{zzG})_{OB} + (M_{OB})(d_{OB})^2$

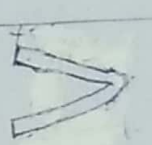
$= 6.51 + (3.125(2.5)^2) = 26.04\ kgm^2$

Mass M.I of composite rod about Z-axis

$(I_{zz}) = \Sigma(I_{zz}) = 26.04 + 71.25 + 26.04 = 123.33\ kgm^2$

K (radius of gyration) $= \sqrt{\frac{I_{zz}}{M}} = \sqrt{\frac{123.33}{10}}$

$= 3.51m$

$d_{OB} = d_{OA} =$  = Distance of OB or OA from origin $= \frac{5}{2} = 2.5 = 5\cos 60°$

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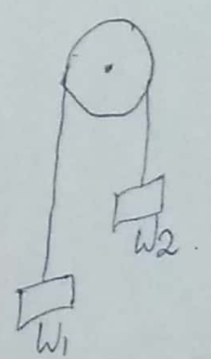
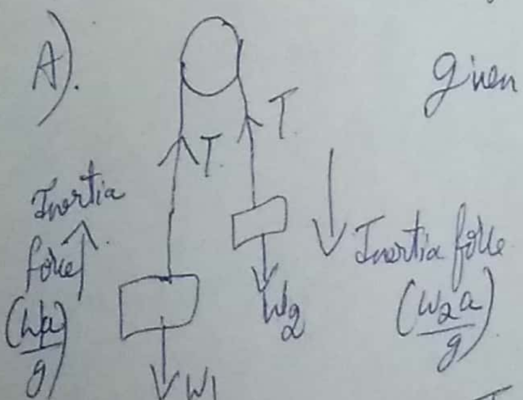
* Two blocks of weight (W1) and (W2) are connected by inextensible wire passing over a smooth pulley. If W1 is greater than W2, find the tension in string and the acceleration of system.



A). given $W_1 > W_2$



$\frac{W_1 a}{g} + T = W_1$

$T = W_1 - \frac{W_1 a}{g}$ — (1)

$T = W_2 + \frac{W_2 a}{g}$ — (2)

$W_2 + \frac{W_2 a}{g} = W_1 - \frac{W_1 a}{g}$

$W_1 - W_2 = \frac{W_2 a + W_1 a}{g}$

$W_1 - W_2 = \frac{a}{g}(W_2 + W_1)$

$a = \frac{(W_1 - W_2)g}{W_2 + W_1}$

$$T = W_2 + \frac{W_2}{g} a \qquad\qquad T = W_2 + \frac{W_1 W_2 - W_2^2}{W_1 + W_2}$$

$$T = W_2 + \frac{W_2}{g}\left(\frac{W_1 - W_2}{W_1 + W_2}\right) g \qquad\qquad T = \frac{W_2 W_1 + W_2^2 + W_1 W_2 - W_2^2}{W_1 + W_2}$$

$$T = \frac{2 W_1 W_2}{W_1 + W_2} \qquad\qquad a = \left(\frac{W_1 - W_2}{W_1 + W_2}\right) g$$

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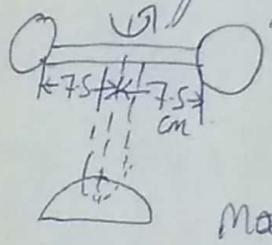
* Determine mass M.I of composite solid shown about the axis of rotation. The solid is made up of two identical spheres each of 2kg mass and 3cm radius attached at ~~slender rod~~ end of slender rod of 400grams and 15cm length



A) mass M.I of slender rod about its centroidal axes

mass of rod = 400 gram = 0.4 Kg

$$I_{1G} = \frac{M_1 l_1^2}{12} = \frac{0.4(0.15)^2}{12} = 7.5 \times 10^{-4}\ Kg\ m^2$$

mass M.I of sphere about its centroidal axes

$$I_{2G} = \frac{2}{5} M_2 R_2^2 = \frac{2}{5} \times (2)(0.03)^2 = 7.2 \times 10^{-4}\ Kg\ m^2$$

mass M.I of sphere about ~~its~~ axis of rotation is obtained by transfer (parallel axes) formula

$$I_2 = I_{2G} + M_2 d^2 = 7.2 \times 10^{-4} + \underset{\text{mass}}{2}\left(0.03 + \frac{0.15}{2}\right)^2$$

$$d = \underset{\text{radius of sphere}}{0.03} + \underset{\frac{1}{2}(\text{length of rod})}{0.075}$$

$I_2 = 0.0228\ Kg\,m^2$

Hence mass M.I of composite solid about axis of rotation is $I = I_1 + 2(I_2)$

$= 7.5\times10^{-4} + 2[0.0228) = 4.64\times10^{-2}\ Kg\,m^2$

* Find mass M I of a hollow sphere w.r.t a diameter if the $\frac{\text{mass}}{\text{Volume}}$ of material is ρ and Outer & inner radii are R_0 and R_i respectively

A). Consider a ring element of radius (r), radial width (dr) and axial width (dz) at a distance z from the Center of sphere. The mass of this element is

$$M = \rho 2\pi r dr dz$$

and its mass M.I about z-axis is

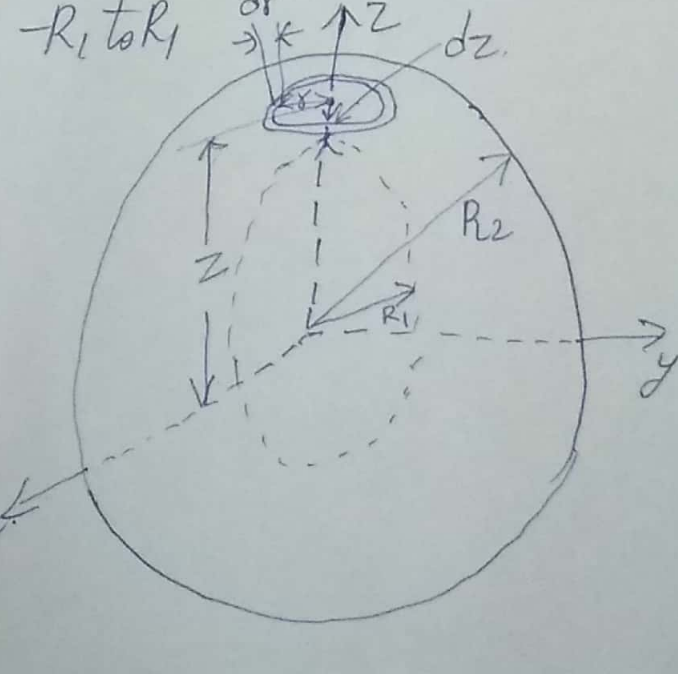
$$dI_{zz} = \rho 2\pi r (dr) dz (r^2) = \rho 2\pi r^3 dr dz.$$

The moment of Inertia of hollow sphere is obtained by Integration

$$\left[\begin{array}{l} r \text{ varying from } 0 \text{ to } \sqrt{(R_2^2 - z^2)} \\ z \text{ varying from } R_1 \text{ to } R_2 \text{ and from } -R_1 \text{ to } -R_2 \end{array}\right]$$

and

$$\left[\begin{array}{l} r \text{ varying from } \sqrt{(R_1^2 - z^2)} \text{ to } +\sqrt{(R_2^2 - z^2)} \\ z \text{ varying from } -R_1 \text{ to } R_1 \end{array}\right]$$



$$I_{zz} = \int_{R_1}^{R_2} \int_{0}^{\sqrt{R_2^2 - z^2}} \rho 2\pi r^3 dr dz$$

$$+ \int_{-R_1}^{-R_2} \int_{0}^{\sqrt{R_2^2 - z^2}} \rho 2\pi r^3 dr dz$$

$$+ \int_{-R_1}^{R_1} \int_{\sqrt{R_1^2 - z^2}}^{\sqrt{R_2 - z^2}} \rho 2\pi r^3 dr dz.$$

$$I_{zz} = \int_{R_1}^{R_2}\int_0^{\sqrt{R_2^2-z^2}} \rho 2\pi r^3 dr\, dz + \int_{-R_2}^{-R_1}\int_0^{\sqrt{R_2^2-z^2}} \rho 2\pi r^3 dr\, dz$$

$$+ \int_{-R_1}^{R_1}\int_{\sqrt{R_1^2-z^2}}^{\sqrt{R_2^2-z^2}} \rho 2\pi r^3 dr\, dz = I_1 + I_2 + I_3$$

Sum of first two ~~integrals~~ integrals is

$$= 2\int_{R_1}^{R_2} \rho 2\pi \frac{(\sqrt{R_2^2-z^2})^4}{4} dz = 2\int_{R_1}^{R_2} \rho 2\pi \frac{(R_2^2-z^2)^2}{4} dz$$

$$= \rho\pi\left[R_2^4 z + \frac{z^5}{5} - 2R_2^2\frac{z^3}{3}\right]_{R_1}^{R_2}$$

$$= \rho\pi\left[R_2^5 + \frac{R_2^5}{5} - \frac{2}{3}R_2^5 - R_2^4 R_1 - \frac{R_1^5}{5} + \frac{2}{3}R_2^2 R_1^3\right)$$

$$= \rho\pi\left(\frac{8}{15}R_2^5 - R_2^4 R_1 - \frac{R_1^5}{5} + \frac{2}{3}R_2^2 R_1^3\right)$$

The third integral is now evaluated as

$$= \int_{-R_1}^{R_1} \rho 2\pi\left[\frac{r^4}{4}\right]_{\sqrt{R_1^2-z^2}}^{\sqrt{R_2^2-z^2}} dz$$

$$= \int_{-R_1}^{R_1} \frac{\rho 2\pi}{4}\left[\left(\sqrt{R_2^2-z^2}\right)^4 - \left(\sqrt{R_1^2-z^2}\right)^4\right] dz$$

$$= \int_{-R_1}^{R_1} \frac{\rho\pi}{2}\left[(R_2^2-z^2)^2 - (R_1^2-z^2)^2\right] dz$$

$$= \frac{\rho\pi}{2}\int_{-R_1}^{R_1}\left(R_2^4 - R_1^4 + 2(R_1^2-R_2^2)z^2\right) dz$$

$$= \frac{\rho\pi}{2}\left[(R_2^4 - R_1^4)z + 2(R_1^2 - R_2^2)\frac{z^3}{3} \right]_{-R_1}^{R_1}$$

$$= \rho\pi \left[R_2^4 R_1 - \frac{2}{3} R_2^2 R_1^3 - \frac{1}{3} R_1^5 \right]$$

Finally adding the three integrals

$$I_{zz} = \rho\pi \left[\frac{8}{15} R_2^5 - R_2^4 R_1 - \frac{1}{5} R_1^5 + \frac{2}{3} R_2^2 R_1^3 + R_2^4 R_1 - \frac{2}{3} R_2^2 R_1^3 - \frac{1}{3} R_1^5 \right]$$

$$I_{zz} = \rho\pi \left(\frac{8}{15} R_2^5 - \frac{8}{15} R_1^5 \right)$$

mass of hollow sphere is

$$M = \frac{4}{3}\pi\rho (R_2^3 - R_1^3) \quad \Big| \quad \underset{\text{(hollow sphere)}}{I_{zz}} = \frac{2}{5} m \frac{(R_2^5 - R_1^5)}{(R_2^3 - R_1^3)}$$

By symmetry about axes and by Interchangeability of the axes

$$\boxed{\underset{\text{of hollow sphere}}{I_{xx} = I_{yy} = I_{zz}} = \frac{2}{5} m \frac{(R_2^5 - R_1^5)}{(R_2^3 - R_1^3)}}$$

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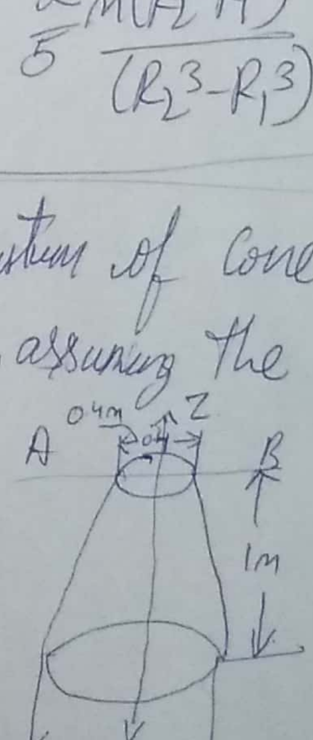
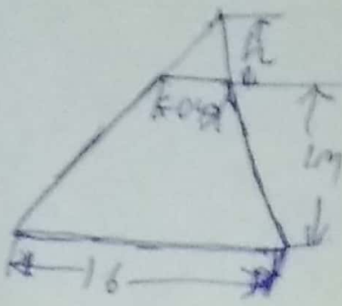
* Calculate the mass M.I of frustum of cone with respect to axis Z-Z and AB, assuming the density of cone is 2500 kg/m^3

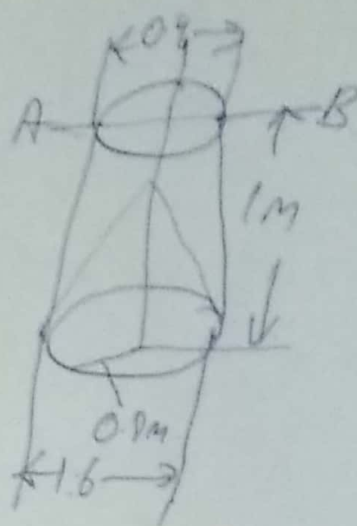
A)



A) Q

Gross section of frustum of cone is shown





Let (h) be height of cone removed

From similar triangles

$$\frac{h+1}{1.6} = \frac{h}{0.4} \quad \Big| \quad h = 0.333m$$

Component (1) $\Rightarrow$ Cone dia = 1.6m and total height of cone $= 1+0.333 = 1.333$

(M_1) Mass $= \rho V = \rho \frac{1}{3}\pi r^2 h$ (Volume of Cone $= \frac{\pi r^2 h}{3}$)

$$= \frac{1}{3}\pi (0.8)^2 \times 1.333 \times 2500 = 2233.46\,Kg$$

Component (2) Cone removed, 0.4 dia and 0.333m height

$$\text{Mass } (M_2) = \frac{1}{3}\pi \underbrace{(0.2)^2}_{r} \underbrace{0.333}_{h} \times \underbrace{2500}_{\rho}$$

$$= 34.87\,Kg$$

Mass M.I about zz axis

$$I_{zz} = (I_{zz})_1 - (I_{zz})_2 = \left(\frac{3}{10} M_1 R_1^2\right) - \left(\frac{3}{10} M_2 R_2^2\right)$$

$$= \frac{3}{10}\left((2233.46 \times 0.8^2) - (34.87 \times 0.2^2)\right) = 428.4\,Kg\,m^2$$

Mass M.I about AB axis

$$(I_{AB})_1 = I_{G1} + M_1 h^2 \qquad \bar{h} = \left\{1 - \left(\frac{1.333}{4}\right)\right\} = 0.67m$$

$$\left(I_{AB}\right)_1 = \left\{\frac{3M_1}{20}\left(\frac{h^2}{4}+R^2\right)\right\} + \left(M_1 \bar{h}^2\right)$$

$$= \left\{\frac{3\times 2233.46}{20}\left(\frac{1.333^2}{4}+0.8^2\right)\right\} + \left\{2233.46\,(0.67)^2\right\}$$

$$\boxed{I_G \text{ of Cone} = \frac{3M}{20}\left(\frac{h^2}{4}+R^2\right)}$$

$$= 1365.83 \text{ Kgm}^2$$

$$\left(I_{AB}\right)_2 = I_{G2} + M_2 \bar{h}^2$$

$$= \left\{\frac{3M_2}{20}\left(\frac{h^2}{4}+R^2\right)\right\} + \left\{M_2\left(\frac{h}{4}\right)^2\right\}$$

$$= \left\{\frac{3\times 34.87}{20}\left(\frac{0.333^2}{4}+0.2^2\right)\right\} + \left\{34.87\left(\frac{0.333}{4}\right)^2\right\}$$

$$= 0.5959 \text{ Kgm}^2$$

$$I_{AB} = 1365.83 - 0.5959 = 1365.23 \text{ Kgm}^2$$